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AXISYMMETRIC SURFACE BUCKLING OF AN ELASTIC HALF-SPACE UNDER COMPRESSION

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The subject of investigation is the axisymmetric surface buckling of a transversally isotropic half-space under compression in a three-dimensional setup. Subcritical strains are assumed to be small and homogeneous.

A half-space, transversally isotropic with respect to the Ox_3 axis, is compressed in two mutually orthogonal directions along the Ox_1 and Ox_2 axes under a load of intensity p. A compressive load of intensity q is applied to the surface $x_3 = 0$ of a half-space.

Let us use the linearized stability equations [1]

$$[\sigma^*_{jm} - p_{jn}u^*_{m,n}]_{,j} = 0,$$

where σ^*_{jm}, and u^*_m are strain and displacement disturbances. When the body is transversally isotropic and the load is given by $p_{11} = p$, $p_{22} = p$, and $p_{33} = q$, the stability equations have the form

$$\begin{gathered}(a_{11}-p)u^*_{1,11} + (G_{12}-p)u^*_{1,22} + (G-q)u^*_{1,33} + (G_{12}+a_{12})u^*_{2,12} + (G+a_{13})u^*_{3,13} = 0,\\ (G_{12}+a_{12})u^*_{1,12} + (G_{12}-p)u^*_{2,11} + (a_{11}-p)u^*_{2,22} + (G-q)u^*_{2,33} + (G+a_{13})u^*_{3,23} = 0,\\ (G+a_{13})u^*_{1,13} + (G+a_{13})u^*_{2,23} + (G-p)(u^*_{3,11}+u^*_{3,22}) + (a_{33}-q)u^*_{3,33} = 0.\end{gathered} \tag{1}$$

By means of the two-dimensional Fourier transform with respect to the x_1 and x_2 coordinates

$$u_j(\zeta,\eta,x_3) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u^*_j(x_1,x_2,x_3)\exp(i(\zeta x_1+\eta x_2))\,dx_1\,dx_2,$$

stability equations (1) are transformed into the following system of ordinary differential equations with respect to the Fourier transforms of displacement disturbances $u_1(\zeta,\eta,x_3)$, $u_2(\zeta,\eta,x_3)$, and $u_3(\zeta,\eta,x_3)$:

$$\begin{gathered}((a_{11}-p)\zeta^2 + (G_{12}-p)\eta^2)u_1 - (G-q)u_{1,33} + \zeta\eta(G_{12}+a_{12})u_2 + i\zeta(G+a_{13})u_{3,3} = 0,\\ (G_{12}+a_{12})\zeta\eta u_1 + ((G_{12}-p)\zeta^2 + (a_{11}-p)\eta^2)u_2 - (G-q)u_{2,33} + i\eta(G+a_{13})u_{3,3} = 0,\\ i(G+a_{13})(\zeta u_{1,3}+\eta u_{2,3}) + (G-p)\rho^2 u_3 - (a_{33}-q)u_{3,33} = 0\end{gathered} \tag{2}$$

($\rho^2 = \zeta^2 + \eta^2$). Under the assumption that $u_2 = (\eta/\zeta)u_1$, system (2) is reduced to two equations:

$$\begin{gathered}\rho^2(a_{11}-p)u_1 - (G-q)u_{1,33} + i\zeta(G+a_{13})u_{3,3} = 0,\\ i(G+a_{13})\rho^2(\zeta)^{-1}u_{1,3} + (G-p)\rho^2u_3 - (a_{33}-q)u_{3,33} = 0.\end{gathered} \tag{3}$$

In turn, the system of equations (3) reduces to the equation of the fourth-order

$$u_{3,3333} - \rho^2 b_1 u_{3,33} + \rho^4 b_2 u_3 = 0. \tag{4}$$

Here

$$\begin{gathered}b_1 = ((G-q)(G-p) - (a_{13}+G)^2 + (a_{11}-p)(a_{33}-q))/((G-q)(a_{33}-q));\\ b_2 = ((a_{11}-p)(G-p))/((G-q)(a_{33}-q)).\end{gathered}$$

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The local buckling of the surface is characterized by fading of displacement disturbances as the distance from the epicenter of the disturbances increases. The solution to differential equation (4) is depth-fading off the surface when $x_3 \to -\infty$ and has the form

$$u_3(\zeta,\eta,x_3) = C_1(\zeta,\eta)\exp(\rho k_1 x_3) + C_2(\zeta,\eta)\exp(\rho\, k_2\, x_3),$$

where $C_1(\zeta,\eta)$ and $C_2(\zeta,\eta)$ are arbitrary functions;

$$k_{1,2} = (0.5\, b_1 \pm (0.25\, b_1^2 - b_2)^{0.5})^{0.5}.$$

Disturbances of displacements also have to fade when $|x_1| \to \infty$, and $|x_2| \to \infty$ which corresponds to fading of their Fourier transforms when $|\zeta| \to \infty$ and $|\eta| \to \infty$. To satisfy this condition, assume that $C_1(\zeta,\eta) = A_1 \exp(-\rho c)$, $C_2(\zeta,\eta) = A_2 \exp(-\rho c)$ (A_1, A_2 are arbitrary constants and c is a positive constant). Finally, we write the Fourier transform of displacement disturbance $u_3(\zeta,\eta,x_3)$ in the form

$$u_3(\zeta,\eta,x_3) = A_1 \exp(\rho(k_1x_3 - c)) + A_2 \exp(\rho(k_2x_3 - c)). \tag{5}$$

Fourier transforms of displacement disturbances $u_1(\zeta,\eta,x_3)$ and $u_2(\zeta,\eta,x_3)$ are determined from system (3):

$$\begin{aligned} u_1(\zeta,\eta,x_3) &= (i\zeta/\rho)(A_1m_1 \exp(\rho(k_1x_3 - c)) + A_2m_2 \exp(\rho(k_2x_3 - c))),\\ u_2(\zeta,\eta,x_3) &= (i\eta/\rho)(A_1m_1 \exp(\rho(k_1x_3 - c)) + A_2m_2 \exp(\rho(k_2x_3 - c))). \end{aligned} \tag{6}$$

Here

$$m_1 = k_1(d_1 - d_2^2k_1); \qquad m_2 = k_2(d_1 - d_2^2k_2);$$
$$d_1 = ((G-p)(G-q) - (a_{13}+G)^2)/((a_{13}+G)(a_{11}-p));$$
$$d_2 = ((a_{33}-q)(G-q))/((a_{13}+G)(a_{11}-p)).$$

We use an inverse Fourier transform to calculate the originals of the displacement disturbances:

$$u_j^*(x_1,x_2,x_3) = \frac{1}{2\pi}\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} u_j(\zeta,\eta,x_3)\exp(i(\zeta x_1 + \eta x_2))\, d\zeta\, d\eta.$$

After computations we get

$$u_1^*(x_1,x_2,x_3) = \frac{A_1m_1x_1}{(r^2 + (k_1x_3 - c)^2)^{3/2}} + \frac{A_2m_2x_1}{(r^2 + (k_2x_3 - c)^2)^{3/2}},$$

$$u_2^*(x_1,x_2,x_3) = \frac{A_1m_1x_2}{(r^2 + (k_1x_3 - c)^2)^{3/2}} + \frac{A_2m_2x_2}{(r^2 + (k_2x_3 - c)^2)^{3/2}},$$

$$u_3^*(x_1,x_2,x_3) = \frac{-A_1(k_1x_3 - c)}{(r^2 + (k_1x_3 - c)^2)^{3/2}} + \frac{-A_2(k_2x_3 - c)}{(r^2 + (k_2x_3 - c)^2)^{3/2}},$$

where $r^2 = x_1^2 + x_2^2$.

To determine the arbitrary constants A_1 and A_2, we employ the boundary conditions [1]

$$P_m^* = n_j(\sigma_{jm}^* - p_{jn}u_{m,n}^*(1 - \delta_{mn})),$$

which on the surface $x_3 = 0$ under the given load $p_{33} = q$ take the following form:

$$P_1^* = G(u_{1,3}^* + u_{3,1}^*) - qu_{1,3}^*, \quad P_2^* = G(u_{2,3}^* + u_{3,2}^*) - qu_{2,3}^*, \quad P_3^* = a_{13}(u_{1,1}^* + u_{2,2}^*) + a_{33}u_{3,3}^*. \tag{7}$$

If the load $p_{33} = q$ on the surface $x_3 = 0$ is a "tracing" one, disturbances $P_1^* = -qu_{3,1}^*$, $P_2^* = -qu_{3,2}^*$, and $P_3^* = 0$ appear. The equality $P_1^* = P_2^* = P_3^* = 0$ holds for "dead" load disturbances. Let us apply a two-dimensional Fourier transform to the boundary conditions (7). In the case of a "dead" load, we have

$$(G-q)u_{1,3} - i\,G\zeta u_3 = 0, \quad (G-q)u_{2,3} - i\,G\eta_3 u = 0, \quad a_{33}u_{3,3} - i\,a_{13}(\zeta u_1 + \eta u_2) = 0. \tag{8}$$

Substitution of the transforms of displacement disturbances (5) and (6) into the boundary conditions (8) yields an algebraic system of linear homogeneous equations with respect to the unknowns A_1 and A_2:

$$A_1((G-q)k_1m_1 - G) + A_2((G-q)k_2m_2 - G) = 0,$$
$$A_1((G-q)k_1m_1 - G) + A_2((G-q)k_2m_2 - G) = 0,$$
$$A_1(a_{13}m_1 + a_{33}k_1) + A_2(a_{13}m_2 + a_{33}k_2) = 0.$$

Since the first two equations coincide, the requirement of the existence of a nonzero solution of two linear equations leads to the characteristic equation for "critical" loads p_0 and q_0:

$$(k_1 - k_2)((G-q)a_{13}m_1m_2 + a_{33}G) + (m_1 - m_2)((G-q)a_{33}k_1k_2 + a_{13}G) = 0.$$

In the case of a "tracing" load, after analogous computations we obtain the characteristic equation

$$(k_1 - k_2)((G-q)a_{13}m_1m_2 + a_{33}(G+q)) + (m_1 - m_2)((G-q)a_{33}k_1k_2 + a_{13}(G+q)) = 0.$$

Computer analysis shows that:

(1) local surface buckling is possible only in media with small shear rigidity $G = G_{13} = G_{23}$ since in other media the critical load p_0 has a very great value and becomes unreal;

(2) the critical load p_0 increases only slightly with the growth of the elastic modulus ratio $E_0 = E/E^*$, where E is Young's modulus in the plane of isotropy x_1Ox_2 and E^* is Young's modulus in the direction of the Ox_3 axis;

(3) the load q on the surface $x_3 = 0$ increases only slightly the critical load p_0 compared to the case of a free surface;

(4) the critical loads p_0 are equal in the case of "dead" and "tracing" loads q on the surface $x_3 = 0$.

REFERENCES

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